Penrith Selective High School

March 2015

Higher School Certificate

Half Yearly Examination.

Extension 1 Mathematics

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided towards the back of this paper
- In Questions 6-9, show relevant mathematical reasoning and/or calculations
- All diagrams are not to scale

Total Marks – 65

Section I Pages 2–3

5 marks

- Attempt Questions 1–5
- Allow about 8 minutes for this section
- Answer these questions on the multiple choice answer sheet provided at the back of this paper

Section II Pages 4–7

60 marks

- Attempt Questions 6–9
- Allow about 2 hours 52 minutes for this section
- Start each question on a new page

Teacher Use Only	Questions 1-5	Question 6	Question 7	Question 8	Question 9	Total	%
Marks	/5	/15	/15	/15	/15	/65	

Student Name: ______________________________

Section I:
5 marks
Attempt Questions 1–5
Allow about 8 minutes for this section
Use the multiple-choice answer sheet for Questions 1–5.

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1. Simplify $\frac{log_b(9a^2)}{log_b(3a)}$, where $a > 0, b > 0$

(A) 2

(B) $3a$

(C) $log_b(3a)$

(D) $log_b(9a^2 - 3a)$

2. If $\alpha, \beta\ and\ \gamma$ are the roots $of\ x^3 - 4x^2 + 2x + 3 = 0$, what is the value of: $\alpha\beta + \beta\gamma + \cdot\gamma\alpha$?

(A) -2

(B) -4

(C) 4

(D) 2

3. If $P(x)$ has degree $n + 1$, then $P(x) = 0$:

(A) always has n turning points

(B) has at most n turning points

(C) always has $n + 1$ turning points

(D) has at most $n + 1$ turning points

Section 1 continued on next page.

4. Which of the following is an expression for $\int cos^2 8x \, dx$?

(A) $\frac{x}{2} - \frac{1}{32} sin8x + c$

(B) $\frac{x}{2} + \frac{1}{32} sin8x + c$

(C) $\frac{x}{2} - \frac{1}{32} sin16x + c$

(D) $\frac{x}{2} + \frac{1}{32} sin16x + c$

5. How many numbers greater than 3000 can be formed with the digits 2, 3, 4, 5 and 6 if no digit is used more than once in the number? (You do not have to use all of the digits).

(A) 96

(B) 120

(C) 196

(D) 216

END OF SECTION I
Section 2 begins on next page.

Section II
60 Marks
Attempt Questions 6–9
Allow about 1 hour and 52 minutes for this section
Start each question on a new page. Extra paper is available.
In Questions 6-9, your responses should include relevant mathematical reasoning and/or calculations.

Question 6 (15 marks) Start a new page. **Marks**

a) Find the acute angle between the two lines whose equations are $9x - 2y = 8$ and $11x + 7y - 12 = 0$. **(3)**

b) i) Expand $(a - x)^3$ **(2)**

ii) Use your expansion to show that $(1.98)^3 = 7.762$ when rounded off to 3 decimal places. **(2)**

c) The polynomial $(x - a)^3 + b$ is zero at $x = 1$ and, when divided by x, the remainder is -7. Find all possible values of a and b. **(4)**

d) *M* and *N* have co-ordinates $(-1,7)$ and $(5,-2)$ respectively. *P* divides MN in the ratio $k: 1$.

i) Find an expression for the co-ordinate of *P*. **(2)**

ii) Hence, find the value of *k* when *P* lies on the line $5x - 4y - 1 = 0$. **(2)**

End of Question 6
Question 7 on next page

Question 7 (15 marks) Start a new page. **Marks**

a) If $y = ln(cosx)$, show that: $\frac{dy}{dx} = -tanx$ **(2)**

b) i) Sketch $y = |2x - 1|$and $y = |2x + 1|$on the same set of axes, showing key features. **(3)**

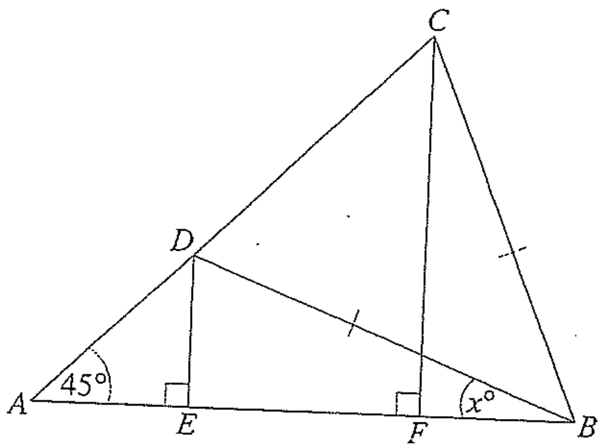
ii) Hence, find all real numbers x, such that $|2x - 1| < |2x + 1|$ **(1)**

c) $P(x_1, y_1)$ is any point on the curve $y = x^n$. PM and PN are perpendiculars to the x and y-axes respectively, with M and N lying on the x and $y -$ axes respectively.

i) Draw a diagram, marking all the information given above. **(1)**

ii) Show that the area enclosed between the curve, the x-axis and PM is $\frac{1}{n+1}$ of the area of the rectangle $OMPN$. (O is the origin). **(3)**

d) $\angle BAC = 45^\circ, DE \perp AB, CF \perp AB, BC = BD, \angle ABD = x^\circ$



Prove that:

i) $\Delta BDE \equiv \Delta CBF$ **(3)**

ii) $AE = FB$ **(2)**

End of Question 7
Question 8 on next page.

Question 8 (15 marks). Start a new page. **Marks**

a) Evaluate $tan\left[cos^{-1}\left(\frac{3}{5}\right)\right]$ **(2)**

b) i) Show that $\frac{x^2+8}{x^2+4}$ can be written as $1+\frac{4}{x^2+4}$ **(1)**

ii) Hence evaluate $\int_0^2 \frac{x^2+8}{x^2+4}dx$ **(2)**

c) i) For what values of x is the function $sin^{-1}x$ defined? **(1)**

ii) Find the maximum value of $2x(1-x)$ **(1)**

iii) Find the range of the function $f(x) = sin^{-1}[2x(1-x)]$ with domain $0 \le x \le 1$ **(2)**

d) The stationary points on the graph of $y = \frac{12x-x^3}{8}$ are given to be $(2,2)$ and $(-2,-2)$. (This can be assumed).

i) A function is defined by $f(x) = \frac{12x-x^3}{8}$, for $-2 \le f(x) \le 2$. Sketch a graph of $y = f(x)$ and explain why an inverse function $f^{-1}(x)$ exists. **(3)**

ii) Sketch $y = f^{-1}(x)$ on the same set of axes. **(1)**

iii) Find the value of $\int_0^2 f^{-1}(x)dx$ **(2)**

End of Question 8
Question 9 on next page

Question 9 (15 marks) Start a new page. **Marks**

a) If $t = tan\frac{x}{2}$, express as simply as possible in terms of t, $\frac{1+\cos x}{1-\cos x}$ (2)

b) i) Show that $8cosx - 15sinx$ can be written as $17cos(x + 61^\circ 56')$. (3)
(Where $61^\circ 56'$ has been rounded off correct to the nearest minute).

ii) Hence, find the maximum value of $8cosx - 15sinx$, and the smallest positive value of x for which this maximum occurs. Give your answer correct to the nearest minute. (2)

c) i) Show that $1 + sin2A$ can be written as $(cosA + sinA)^2$ (2)

ii) Prove that $\sqrt{\frac{1-sin2x}{1+sin2x}} = \frac{1-tanx}{1+tanx}$ (3)

iii) Hence, show that the exact value of $tan\frac{\pi}{8} = \sqrt{2} - 1$ (3)

End of Paper

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax\tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Student Name:____

Multiple Choice Answer Sheet

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample:	2 + 4 =	(A) 2	(B) 6	(C) 8	(D) 9
		A ○	B ●	C ○	D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ⊗ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word 'correct' and drawing an arrow as follows.

A ⊗ B ⊗ (correct) C ○ D ○

Start here ⟶	**1.**	A ○	B ○	C ○	D ○
	2.	A ○	B ○	C ○	D ○
	3.	A ○	B ○	C ○	D ○
	4.	A ○	B ○	C ○	D ○
	5.	A ○	B ○	C ○	D ○

Suggested Solutions	Marker's Comments

1 A 2 D 3 B 4 D 5 D

Q6. a) $9x - 2y = 8$

$y = \frac{9x-8}{2}$

$m_1 = \frac{9}{2}$

$11x + 7y - 12 = 0$

$y = \frac{-11x+12}{7}$

$m_2 = -\frac{11}{7}$

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

(Learn this formula)

$\tan\theta = \left|\frac{\frac{9}{2} - -\frac{11}{7}}{1 - \frac{9}{2}\times\frac{11}{7}}\right|$

$= |1|$

$\theta = \tan^{-1}(1) = 45°$ or $\frac{\pi}{4}$

b) i) $(a-x)^3 = a^3 - 3a^2x + 3ax^2 - x^3$

(ii) $(1.98)^3 = (2 - 0.02)^3$

$a = 2,\ x = 0.02$

$(1.98)^3 = 2^3 - 3\times 2^2 \times 0.02$
$+ 3\times 2\times 0.02^2 - 0.02^3$

$= 8 - 0.24 + 0.0024 - 0.02^3$

$= 7.762392$ (Put this line in) ⟵

$= 7.762$ (to 3 dp)

Exam	MATHEMATICS : Question......	
	Suggested Solutions	Marker's Comments

6 c) $P(1) = 0$

$\underline{(1-a)^3 + b = 0}$

$1 - 3a + 3a^2 - a^3 + b = 0$ ①

$P(0) = -7$ (Remainder Theorem) ⟸

Many students forgot the remainder theorem

$\underline{(-a)^3 + b = -7}$ ②

Sub ② into ①

$1 - 3a + 3a^2 - 7 = 0$

$3a^2 - 3a - 6 = 0$

$a^2 - a - 2 = 0$

$(a-2)(a+1) = 0$

$\underline{a = 2, \quad b = -7 + 8 = 1}$

or $\underline{a = -1, \quad b = -7 + -1 = -8}$

Exam	MATHEMATICS : Question......	
	Suggested Solutions	Marker's Comments

(i) M $(-1, 7)$ $x_1\ y_1$ N $(5, -2)$ $x_2\ y_2$ $P(x, y)$

$k : 1$

$m : n$

$$x = \frac{nx_1 + mx_2}{m+n} = \frac{1 \times -1 + k \times 5}{k+1}$$

$$x = \frac{5k-1}{k+1}$$

[Again, students were not remembering formula correctly]

$$y = \frac{1 \times 7 + k \times -2}{k+1} = \frac{7-2k}{k+1}$$

$$P\left(\frac{5k-1}{k+1}, \frac{7-2k}{k+1}\right)$$

(ii) $5x - 4y - 1 = 0$

Sub P into this

$$5\left(\frac{5k-1}{k+1}\right) - 4\left(\frac{7-2k}{k+1}\right) - 1 = 0$$

$$\frac{25k - 5 - 28 + 8k - k - 1}{k+1} = 0$$

$$32k - 34 = 0$$

$$k = \boxed{\frac{17}{16}}$$

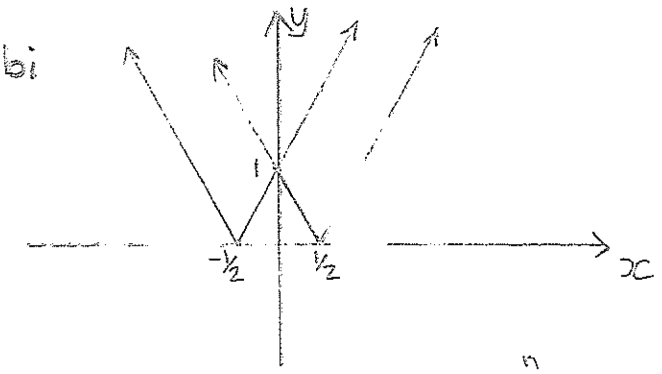
Exam	MATHEMATICS	: Question...... Seven

Suggested Solutions	Marker's Comments

a. $\frac{dy}{dx} = \frac{-\sin x}{\cos x}$

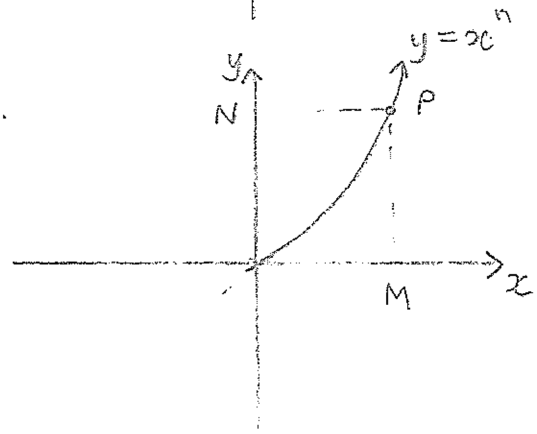
$= -\tan x$

bi



ii. from graph $x > 0$

c.



Area under curve :

$A = \int_0^M x^n \, dx$

$= \left[\frac{x^{n+1}}{n+1} \right]_0^M = \frac{M^{n+1}}{n+1}$

graph goes through origin ✓

Area rectangle $= M \times N \quad (N = M^n)$

$= M \times M^n$

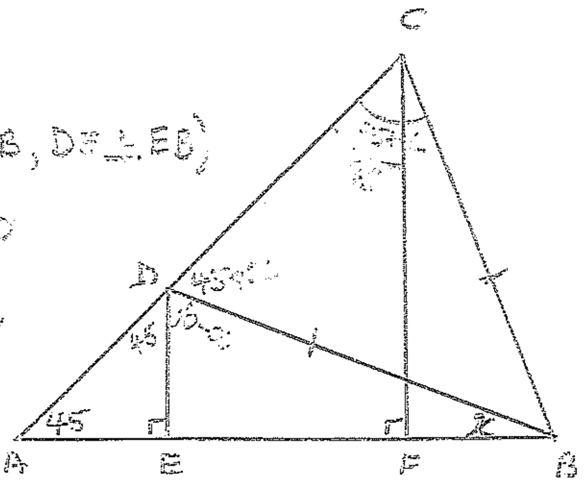
$= M^{n+1}$

Area rectangle $\times \frac{1}{n+1} = \frac{M^{n+1}}{n+1}$

$=$ Area under curve

Exam	MATHEMATICS	: Question...... [illegible]

Suggested Solutions	Marker's Comments



d. i [illegible] $= \angle CBF$

$CB =$ [illegible]

$\angle CFB = \angle$ [illegible] $= 90^\circ$ (C[illegible]$\perp FB$, D[illegible]$\perp EB$)

$\angle BDE =$ [illegible] ($\angle$ sum $\triangle$[illegible])

$\angle FDA =$ [illegible] $\angle$[illegible]A)

$\angle CDB = 180 - (45 +$ [illegible]$)$

$= 45$ [illegible]

(adjacent supplementary [illegible])

$\triangle BCD$ is isosceles ($BD =$ [illegible])

[illegible] $= \angle CDB =$ [illegible] (equal angles [illegible])

$\angle DC$[illegible] $= \angle$[illegible] (corresponding angles [illegible])

$\therefore \angle BC$[illegible] $= x$ ([illegible] of adjacent angles)

$\therefore \angle$[illegible] $=$ [illegible]

$\therefore \triangle BD$[illegible] $\equiv \triangle CBF$ (AAS)

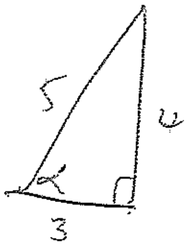
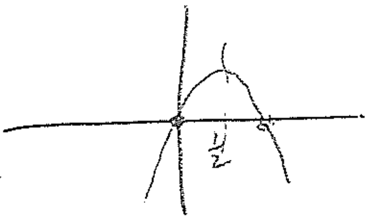
ii AE = [illegible] (equal [illegible] of [illegible])

DE = FB (corresponding [illegible] of [illegible])

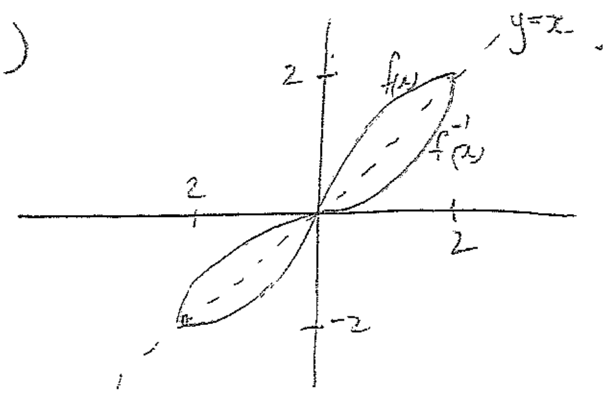
$\therefore AE = FB$.

Marker's Comments: Students [illegible] to write reasons for each line of proof.

Exam Ext 1 Half Yearly - MATHEMATICS : Question 8

Suggested Solutions	Marker's Comments
a) $\cos^{-1}\frac{3}{5} = \alpha$  $\tan\alpha = \frac{4}{3}$	well done.
b)(i) $\frac{x^2+8}{x^2+4} = \frac{x^2+4+4}{x^2+4} = \frac{x^2+4}{x^2+4} + \frac{4}{x^2+4} = 1 + \frac{4}{x^2+4}$	well done
(ii) $\int_0^2 1 + \frac{4}{x^2+4}$ $= [x]_0^2 + 2\left[\tan^{-1}\frac{x}{2}\right]_0^2$ $= 2 + \frac{\pi}{2}$.	well done except some students used wrong method to integrate $\frac{4}{x^2+4}$
c(i) $-1 \leq x \leq 1$	well done
(ii) $2x(1-x)$  $f(\frac{1}{2}) = 1 \times \frac{1}{2} = \frac{1}{2}$ ∴ max value is $\frac{1}{2}$.	students who used calculus create a lot of work
(iii) $\sin^{-1}(\frac{1}{2}) = \frac{\pi}{6}$ ∴ max $\sin^{-1}[2x(1-x)]$ can reach. Range $0 \leq 2x(1-x) \leq \frac{1}{2}$, $\sin^{-1}(0) =$ $0 \leq y \leq \frac{\pi}{6}$	Students who didn't understand concept got this wrong

Exam Ext 1 Half Yearly MATHEMATICS : Question 8

Suggested Solutions	Marker's Comments
(d) (i)  each y value of f(x) corresponds to a unique x value.	Graphs generally needed to be larger and clearer.
(iii) $\int_0^2 f^{-1}(x)\,dx \ldots$	students used a variety of methods to find area. Some only got part way.

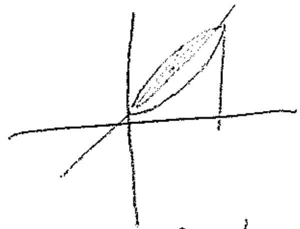
$\int_0^2 \frac{12x - x^3}{8}\,dx$ $\qquad$ $\int_0^2 x\,dx$

$\frac{1}{8}\left[6x^2 - \frac{x^4}{4}\right]_0^2$ $\qquad$ $\left[\frac{x^2}{2}\right]_0^2$

$= \frac{1}{8}\left[\left(6(4) - \frac{16}{4}\right) - 0\right]$ $\qquad$ $= 2$

$= \frac{1}{8}(20)$

$= \frac{20}{8}$



$\frac{20}{8} - 2 = \frac{1}{2}$ (the area of shaded bit)

$\therefore 2 - \frac{1}{2} = 1\frac{1}{2}$ unit^2

since the reflection of this area around $y = x$ is the area we need to subtract from triangle area.

Exam Ext 1 Half Yearly	MATHEMATICS	: Question 9

Suggested Solutions	Marker's Comments
a) $\tan\frac{x}{2}$ $\frac{1+\frac{1-t^2}{1+t^2}}{1-\frac{1-t^2}{1+t^2}} = \frac{1+t^2+1-t^2}{1+t^2} \div \frac{1+t^2-1+t^2}{1+t^2}$ $= \frac{2}{2t^2}$ $= \frac{1}{t^2}$	well done some algebraic errors
b(i) $8\cos x - 15\sin x \equiv A\cos(x+\alpha)$ $\equiv A(\cos x\cos\alpha - \sin x\sin\alpha)$ $\equiv A\cos x\cos\alpha - A\sin x\sin\alpha$ $A\cos\alpha = 8$ $A\sin\alpha = 15$ $\tan\alpha = \frac{15}{8}$ $\alpha = 61°56'$ $A^2 = 8^2 + 15^2$ $A = 17$ $\therefore 8\cos x - 15\sin x \equiv 17\cos(x + 61°56')$	well done
(ii) Max value = 17 (since max value of cos is 1) this happens when $\cos(x+61°56') = 1$ i.e. when $x + 61°56' = 0$ or 360 $\therefore x = -61°56'$ or $298°4'$ $\therefore x = 298°4'$ (smallest positive value)	well done

Exam Ext I Half yearly MATHEMATICS : Question 14	
Suggested Solutions	Marker's Comments

c(i) $1+\sin 2A = (\cos A+\sin A)^2$

R.H.S. $= \cos^2 A + 2\sin A\cos A + \sin^2 A$

$= 1 + 2\sin A\cos A$

$= 1 + \sin 2A.$

well done

(ii) L.H.S $= \sqrt{\dfrac{1-\sin 2x}{1+\sin 2x}}$

$= \sqrt{\dfrac{(\cos x-\sin x)^2}{(\cos x+\sin x)^2}}$ [using part (i)]

$= \dfrac{\cos x-\sin x}{\cos x+\sin x}$

$= \dfrac{\frac{\cos x}{\cos x} - \frac{\sin x}{\cos x}}{\frac{\cos x}{\cos x} + \frac{\sin x}{\cos x}}$ [÷ top and bottom by $\cos A$]

$= \dfrac{1-\tan x}{1+\tan x}$

students who failed to see how to use part (i) created a lot of work. and generally failed.

(iii) R.H.S $= \dfrac{1-\tan x}{1+\tan x}$

$\tan\left(\frac{\pi}{4}-\frac{\pi}{8}\right) = \dfrac{\tan\frac{\pi}{4}-\tan\frac{\pi}{8}}{1+\tan\frac{\pi}{4}\tan\frac{\pi}{8}}$

$\tan\frac{\pi}{8} = \dfrac{1-\tan\frac{\pi}{8}}{1+\tan\frac{\pi}{8}}$

let $x = \frac{\pi}{8}$

L.H.S. $= \sqrt{\dfrac{1-\sin\frac{\pi}{4}}{1+\sin\frac{\pi}{4}}}$

$= \sqrt{\dfrac{1-\frac{1}{\sqrt{2}}}{1+\frac{1}{\sqrt{2}}}}$

$= \sqrt{\dfrac{\sqrt{2}-1}{\sqrt{2}+1}}$

$= \sqrt{\dfrac{(\sqrt{2}-1)^2}{1}}$

$= \sqrt{2}-1$

$= \tan\frac{\pi}{8}.$

students needed to show left hand side